KEY

8.2 - Area, Surface of Revolution

Calculus II

1. If the infinite curve $y = e^{-7x}, x \geq 0$, is rotated about the x-axis, find the area of the resulting surface.

$S = \int_0^{\infty} 2\pi y\sqrt{1+(dy/dx)^2}\, dx = 2\pi \int_0^{\infty} e^{-7x}\sqrt{1+(-7e^{-7x})^2}\, dx$

$u = -7e^{-7x}$

$du = 49e^{-7x}dx$

$\to \frac{1}{49}\int \sqrt{1+u^2}\, du = 1/98\, u\sqrt{1+u^2} + \frac{1}{98}\ln(u+\sqrt{1+u^2}) + C$

$= 1/98(-7e^{-x})\sqrt{1+49e^{-14x}} + \frac{1}{98}\ln|-7e^{-x} + \sqrt{1+49e^{-14x}} + C$

$S = 2\pi \lim_{t\to\infty} \frac{1}{49}\int_0^t e^{-7x}\sqrt{1+(-7e^{-7x})^2}\, dx$

$= 2\pi \lim_{t\to\infty} \frac{1}{49}\left[\frac{1}{2}(-7e^{-7x})\sqrt{1+49e^{-14x}} + \frac{1}{2}\ln(-7e^{-7x} + \sqrt{1+49e^{-14x}}\right]_0^t$

$= 1/49\,\pi[7\sqrt{50} - \ln(\sqrt{50}-7)]$

2. Set up the integral for the area of the surface obtained by rotating the given curve about the specified axis, then solve.
$xy = 5y^2 - 1, 1 \leq y \leq 4$, x-axis

$xy = 5y^2 - 1$

$x = 5y - y^{-1}$

$dx/dy = 5 + y^{-2}$

$\to \sqrt{1+(5+y^{-2})^2}\, dy$

$S = \int_1^4 2\pi y\sqrt{1+(5+y^{-2})^2}\, dy \approx 248.8369$

$u = \sin(1/8x)$

$du = 1/8 \cos(1/8x)dx$

3. Find the exact area of the surface obtained by rotating the curve about the x-axis.

$y = \cos(\frac{1}{8}x), 0 \le x \le 4\pi$

$y = \cos(1/8x) \rightarrow y' = -1/8 \sin(1/8x)$

$S = \int_0^{4\pi} 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^{4\pi} \cos(1/8x)\sqrt{1+\frac{1}{64}\sin^2(1/8x)}\,dx$

$= 2\pi\int_0^1 \sqrt{1+\frac{1}{64}u^2}\,8du = 2\pi\int_0^1 \sqrt{64+u^2}\,du$

$= 2\pi\left[u/2\sqrt{64+u^2} + 32\ln(u+\sqrt{64+u^2})\right]_0^1$

$= \pi\sqrt{65} + 64\pi\ln\left(\frac{1+\sqrt{65}}{8}\right)$

4. The given curve is rotated about the x-axis. Set up, but do not evaluate, an integral for the area of the resulting surface.

$x = \ln(6y+1), 0 \le y \le 1$

a. Integrate with respect to x.

$x = \ln(6y+1) \rightarrow e^x = 6y+1 \rightarrow y = 1/6 e^x - 1/6$

$ds = \sqrt{1 + 1/36 e^{2x}}\,dx \rightarrow \int_0^{\ln(7)} 2\pi\left(\frac{1}{6}e^x - \frac{1}{6}\right)\sqrt{1+\frac{1}{36}e^{2x}}\,dx$

$= \int_0^{\ln(7)} \pi\left(\frac{1}{3}e^x - \frac{1}{3}\right)\sqrt{1+\frac{1}{36}e^{2x}}\,dx$

$y = 0 \rightarrow x = 0$

$y = 1 \rightarrow x = \ln(7)$

b. Integrate with respect to y.

$x = \ln(6y+1) \rightarrow dx/dy = 6/6y+1$

$ds = \sqrt{1 + 36/(6y+1)^2}\,dy$

$S = \int_0^1 2\pi y\sqrt{1+36/(6y+1)^2}\,dy$