**Step 3**

$$\sqrt{\mathbf{e}^{8x} - 9} = \sqrt{(3\sec(\theta))^2 - 9} = \sqrt{9\sec^2(\theta) - 9}$$
$$= 3\sqrt{\sec^2(\theta) - 1} = 3\sqrt{\tan^2(\theta)} = \boxed{3\left|\tan(\theta)\right|}$$

Note that because we don't know the values of $\theta$ we can't determine if the tangent is positive or negative and so cannot get rid of the absolute value bars here.

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9. Use a trig substitution to evaluate $\displaystyle\int \frac{\sqrt{x^2+16}}{x^4}\,dx$.

**Step 1**

In this case it looks like we'll need the following trig substitution.

$$x = 4\tan(\theta)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

---

**Step 2**

Let's first use the substitution to eliminate the root.

$$\sqrt{x^2+16} = \sqrt{16\tan^2(\theta) + 16} = 4\sqrt{\sec^2(\theta)} = 4\left|\sec(\theta)\right|$$

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

$$\sqrt{x^2+16} = 4\sec(\theta)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dx = 4\sec^2(\theta)\,d\theta$$

---

## Step 3

Now let's do the actual substitution.

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = \int \frac{4\sec(\theta)}{(4\tan(\theta))^4} 4\sec^2(\theta)\,d\theta = \int \frac{\sec^3(\theta)}{16\tan^4(\theta)}\,d\theta$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

---

## Step 4

We now need to evaluate the integral. In this case the integral looks to be a little difficult to do in terms of secants and tangents so let's convert the integrand to sines and cosines and see what we get. Doing this gives,

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = \frac{1}{16}\int \frac{\cos(\theta)}{\sin^4(\theta)}\,d\theta$$

This is a simple integral to evaluate so here is the integral evaluation.

$$\begin{aligned}\int \frac{\sqrt{x^2+16}}{x^4}\,dx &= \frac{1}{16}\int \frac{\cos(\theta)}{\sin^4(\theta)}\,d\theta \qquad\qquad u=\sin(\theta)\\ &= \frac{1}{16}\int u^{-4}\,du\\ &= -\frac{1}{48}u^{-3}+c = -\frac{1}{48}[\sin(\theta)]^{-3}+c = -\frac{1}{48}\csc^3(\theta)+c\end{aligned}$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

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## Step 5

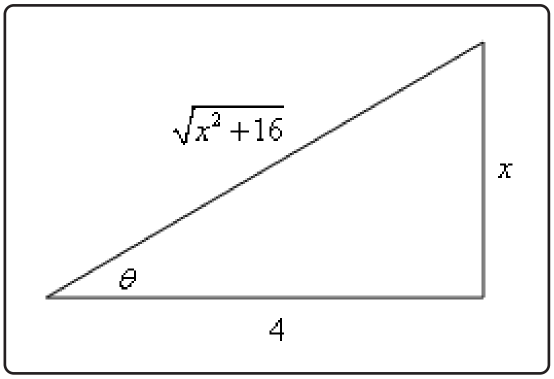
As the final step we just need to go back to $x$'s. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\tan(\theta) = \frac{x}{4} \quad \left(= \frac{\text{opp}}{\text{adj}}\right)$$

From the right triangle we get,

$$\csc(\theta) = \frac{\sqrt{x^2+16}}{x}$$




The integral is then,

$$\int \frac{\sqrt{x^2+16}}{x^4}\,dx = -\frac{1}{48}\left[\frac{\sqrt{x^2+16}}{x}\right]^3 + c = \boxed{-\frac{\left(x^2+16\right)^{\frac{3}{2}}}{48x^3} + c}$$

10. Use a trig substitution to evaluate $\int \sqrt{1-7w^2}\,dw$.

## Step 1

In this case it looks like we'll need the following trig substitution.

$$w = \frac{1}{\sqrt{7}}\sin(\theta)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

## Step 2

Let's first use the substitution to eliminate the root.

$$\sqrt{1-7w^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = |\cos(\theta)|$$

Next, because we are doing an indefinite integral we will assume that the cosine is positive and so we can drop the absolute value bars to get,

$$\sqrt{1-7w^2} = \cos(\theta)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$dw = \frac{1}{\sqrt{7}}\cos\left(\theta\right)\,d\theta$$

## Step 3

Now let's do the actual substitution.

$$\int \sqrt{1-7w^2}\,dw = \int \cos\left(\theta\right)\left(\frac{1}{\sqrt{7}}\cos\left(\theta\right)\right)d\theta = \frac{1}{\sqrt{7}}\int \cos^2\left(\theta\right)\,d\theta$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

## Step 4

We now need to evaluate the integral. Here is that work.

$$\int \sqrt{1-7w^2}\,dw = \frac{1}{\sqrt{7}}\int \frac{1}{2}\left[1+\cos\left(2\theta\right)\right]\,d\theta = \frac{1}{2\sqrt{7}}\left[\theta + \frac{1}{2}\sin\left(2\theta\right)\right] + c$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

## Step 5

As the final step we just need to go back to $w$'s.

To eliminate the the first term (*i.e.* the $\theta$) we can use any of the inverse trig functions. The easiest is to probably just use the original substitution and get a formula involving inverse sine but any of the six trig functions could be used if we wanted to. Using the substitution gives us,

$$\sin\left(\theta\right) = \sqrt{7}\,w \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(\sqrt{7}\,w\right)$$

Eliminating the $\sin\left(2\theta\right)$ requires a little more work. We can't just use a right triangle as we normally would because that would only give trig functions with an argument of $\theta$ and we have an argument of $2\theta$. However, we could use the double angle formula for sine to reduce this to trig functions with arguments of $\theta$ . Doing this gives,

$$\int \sqrt{1-7w^2}\,dw = \frac{1}{2\sqrt{7}}\left[\theta + \sin(\theta)\cos(\theta)\right] + c$$

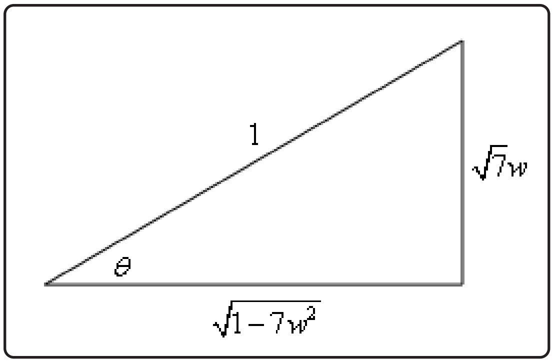
We can now do the right triangle work.

From the substitution we have,

$$\sin(\theta) = \frac{\sqrt{7}w}{1} \qquad \left(=\frac{\text{opp}}{\text{hyp}}\right)$$

From the right triangle we get,

$$\cos(\theta) = \sqrt{1-7w^2}$$




The integral is then,

$$\int \sqrt{1-7w^2}\,dw = \frac{1}{2\sqrt{7}}\left[\sin^{-1}\left(\sqrt{7}w\right) + \sqrt{7}\,w\sqrt{1-7w^2}\right] + c$$

11. Use a trig substitution to evaluate $\int t^3\left(3t^2-4\right)^{\frac{5}{2}}\,dt$.

## Step 1

First, do not get excited about the exponent in the integrand. These types of problems work exactly the same as those with just a root (as opposed to this case in which we have a root to a power - you do agree that is what we have right?). So, in this case it looks like we'll need the following trig substitution.

$$t = \frac{2}{\sqrt{3}}\sec(\theta)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Also notice that upon doing the substation we replaced the $y$ limits with the $\theta$ limits. This will help with a later step.

## Step 5

We now need to evaluate the integral. Here is that work.

$$\begin{aligned}\int_{1}^{4} 2z^5\sqrt{2+9z^2}\,dz &= \frac{16\sqrt{2}}{729}\int_{1.1303}^{1.4535}\left[\sec^2(\theta)-1\right]^2\sec^2(\theta)\;\tan(\theta)\sec(\theta)\,d\theta \\ &= \frac{16\sqrt{2}}{729}\int_{\sec(1.1303)}^{\sec(1.4535)}\left[u^2-1\right]^2u^2du \qquad u=\sec(\theta) \\ &= \frac{16\sqrt{2}}{729}\int_{2.3452}^{8.5440}u^6-2u^4+u^2du \\ &= \frac{16\sqrt{2}}{729}\left[\frac{1}{7}u^7-\frac{2}{5}u^5+\frac{1}{3}u^3\right]\Bigg|_{2.3452}^{8.5440} = \boxed{14178.20559}\end{aligned}$$

Don't forget all the "standard" manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don't recall them you'll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Also, note that because we converted the limits at every substitution into limits for the "new" variable we did not need to do any back substitution work on our answer!

14. Use a trig substitution to evaluate $\displaystyle\int \frac{1}{\sqrt{9x^2-36x+37}}\,dx$.

## Step 1

The first thing we'll need to do here is complete the square on the polynomial to get this into a form we can use a trig substitution on.

$$\begin{aligned}9x^2-36x+37 &= 9\left(x^2-4x+\frac{37}{9}\right)=9\left(x^2-4x+4-4+\frac{37}{9}\right)=9\left[(x-2)^2+\frac{1}{9}\right] \\ &= 9(x-2)^2+1\end{aligned}$$

The integral is now,

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\, dx = \int \frac{1}{\sqrt{9(x-2)^2 + 1}}\, dx$$

Now we can proceed with the trig substitution.

## Step 2

It looks like we'll need to the following trig substitution.

$$x - 2 = \frac{1}{3}\tan(\theta)$$

Next let's eliminate the root.

$$\sqrt{9(x-2)^2 + 1} = \sqrt{\tan(\theta)^2 + 1} = \sqrt{\sec^2(\theta)} = |\sec(\theta)|$$

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

$$\sqrt{9(x-2)^2 + 1} = \sec(\theta)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$(1)\, dx = \frac{1}{3}\sec^2(\theta)\, d\theta \qquad \Rightarrow \qquad dx = \frac{1}{3}\sec^2(\theta)\, d\theta$$

Recall that all we really need to do here is compute the differential for both the right and left sides of the substitution.

## Step 3

Now let's do the actual substitution.

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\, dx = \int \frac{1}{\sec(\theta)}\left(\frac{1}{3}\sec^2(\theta)\right) d\theta = \frac{1}{3}\int \sec(\theta)\, d\theta$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

## Step 4

We now need to evaluate the integral. Here is that work.

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\, dx = \frac{1}{3} \ln\left|\sec(\theta) + \tan(\theta)\right| + c$$

Note that this was one of the few trig substitution integrals that didn't really require a lot of manipulation of trig functions to completely evaluate. All we had to really do here was use the fact that we determined the integral of $\sec(\theta)$ in the previous section and reuse that result here.

## Step 5

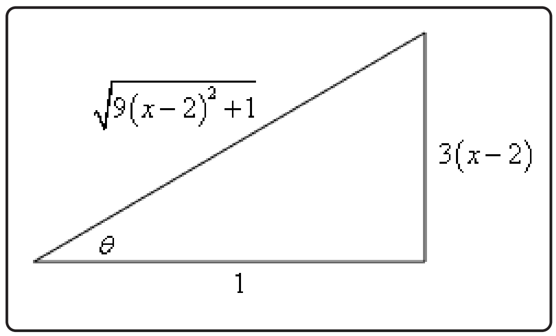
As the final step we just need to go back to $x$'s. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\tan(\theta) = \frac{3(x-2)}{1} \qquad \left(= \frac{\text{opp}}{\text{adj}}\right)$$

From the right triangle we get,

$$\sec(\theta) = \sqrt{9(x-2)^2 + 1}$$




The integral is then,

$$\int \frac{1}{\sqrt{9x^2 - 36x + 37}}\, dx = \boxed{\frac{1}{3} \ln\left|\sqrt{9(x-2)^2 + 1} + 3(x-2)\right| + c}$$

15. Use a trig substitution to evaluate $\displaystyle\int \frac{(z+3)^5}{(40 - 6z - z^2)^{\frac{3}{2}}}\, dz$.

## Step 1

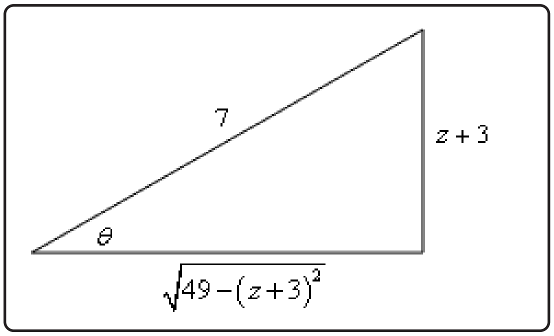
The first thing we'll need to do here is complete the square on the polynomial to get this

From the substitution we have,

$$\sin(\theta) = \frac{z+3}{7} \qquad \left(= \frac{\text{opp}}{\text{hyp}}\right)$$

From the right triangle we get,

$$\cos(\theta) = \frac{\sqrt{49-(z+3)^2}}{7}$$




The integral is then,

$$\int \frac{(z+3)^5}{(40-6z-z^2)^{\frac{3}{2}}}\,dz$$

$$= \boxed{\frac{2401}{\sqrt{49-(z+3)^2}} + 98\sqrt{49-(z+3)^2} - \frac{\left(49-(z+3)^2\right)^{\frac{3}{2}}}{3} + c}$$

---

16. Use a trig substitution to evaluate $\int \cos(x)\sqrt{9+25\sin^2(x)}\,dx$.

**Step 1**

Let's first rewrite the integral a little bit.

$$\int \cos(x)\sqrt{9+25\sin^2(x)}\,dx = \int \cos(x)\sqrt{9+25[\sin(x)]^2}\,dx$$

**Step 2**

With the integral written as it is in the first step we can now see that we do have a sum of a number and something squared under the root. We know from the problems done previously in this section that looks like a tangent substitution. So, let's use the following substitution.

$$\sin(x) = \frac{3}{5}\tan(\theta)$$

Do not get excited about the fact that we are substituting one trig function for another. That will happen on occasion with these kinds of problems. Note however, that we need to be careful and make sure that we also change the variable from $x$ (*i.e.* the variable in the original trig function) into $\theta$ (*i.e.* the variable in the new trig function).

Next let's eliminate the root.

$$\sqrt{9+25[\sin(x)]^2} = \sqrt{9+25\left[\frac{3}{5}\tan(\theta)\right]^2} = \sqrt{9+9\tan^2(\theta)} = 3\sqrt{\sec^2(\theta)} = 3\left|\sec(\theta)\right|$$

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

$$\sqrt{9+25[\sin(x)]^2} = 3\sec(\theta)$$

For a final substitution preparation step let's also compute the differential so we don't forget to use that in the substitution!

$$\cos(x)\,dx = \frac{3}{5}\sec^2(\theta)\,d\theta$$

Recall that all we really need to do here is compute the differential for both the right and left sides of the substitution.

## Step 3

Now let's do the actual substitution.

$$\begin{aligned}\int \cos(x)\sqrt{9+25\sin^2(x)}\,dx &= \int \sqrt{9+25[\sin(x)]^2}\;\cos(x)\,dx \\ &= \int (3\sec(\theta))\left(\frac{3}{5}\sec^2(\theta)\right)d\theta = \frac{9}{5}\int \sec^3(\theta)\,d\theta\end{aligned}$$

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

## Step 4

We now need to evaluate the integral. Here is that work.

$$\int \cos(x)\sqrt{9+25\sin^2(x)}\,dx = \frac{9}{10}\left[\sec(\theta)\tan(\theta) + \ln\left|\sec(\theta)+\tan(\theta)\right|\right] + c$$

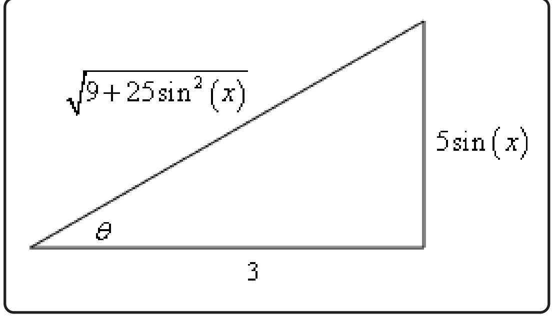
Note that this was one of the few trig substitution integrals that didn't really require a lot of manipulation of trig functions to completely evaluate. All we had to really do here was use the fact that we determined the integral of $\sec^3(\theta)$ in the previous section and reuse that result here.

## Step 5

As the final step we just need to go back to $x$'s. To do this we'll need a quick right triangle. Here is that work.

From the substitution we have,

$$\tan(\theta) = \frac{5\sin(x)}{3} \qquad \left(= \frac{\text{opp}}{\text{adj}}\right)$$




From the right triangle we get,

$$\sec(\theta) = \frac{\sqrt{9+25\sin^2(x)}}{3}$$

The integral is then,

$$\int \cos(x)\sqrt{9+25\sin^2(x)}\,dx = \boxed{\frac{\sin(x)\sqrt{9+25\sin^2(x)}}{2} + \frac{9}{10}\ln\left|\frac{5\sin(x)+\sqrt{9+25\sin^2(x)}}{3}\right| + c}$$