KEY

11.10 - Taylor and Maclaurin Series

Calculus II

1a. Find the Taylor series for f centered at 8 if $f^{(n)}(8) = \frac{(-1)^n n!}{5^n(n+4)}$

$$\sum_{n=0}^{\infty} \frac{f^n(8)}{n!}(x-8)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{5^n(n+4)n!}(x-8)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-8)^n}{5^n(n+4)}$$

1b. What is the radius of convergence R of the Taylor series?

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-8)^{n+1}}{5^{n+1}(n+5)} \cdot \frac{5^n(n+4)}{(-1)^n(x-8)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(x-8)(n+4)}{5(n+5)}\right|$$

$$= \frac{1}{5}|x-8| \lim_{n\to\infty} \frac{n+4}{n+5} = \frac{1}{5}|x-8| \to \frac{1}{5}|x-8| < 1 \to R=5$$

$$|x-8| < 5$$

2a. Find the Maclaurin series for f(x) using definition of a Maclaurin series.

$f(x) = 3(1-x)^{-2}$ $= f(0) + f'(0) + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots$

$$= 3 + 6x + \frac{18}{2}x^2 + \frac{72}{6}x^3 + \ldots$$

$$= 3 + 6x + 9x^2 + 12x^3$$

$$= 3\sum_{n=0}^{\infty}(n+1)x^n$$

2b. Find the associated radius of convergence R.

$$\lim_{n\to\infty}\left|\frac{3(n+2)x^{n+1}}{3(n+1)x^n}\right| = |x| \lim_{n\to\infty} \frac{n+2}{n+1} = |x|(1) = |x| < 1$$

$$\to R=1$$

3a. **Find the Maclaurin series of f(x) using definition of a Maclaurin series.**

$f(x) = cos(x) = f(0) + f'(0) + \frac{f''(0)x^2}{2!} + \frac{f^3(0)x^3}{3!} + \frac{f^4(0)x^4}{4!} + \ldots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

3b. **Find the associated radius of convergence R.**

$\lim_{n \to \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = \lim_{n \to \infty} \frac{x^2}{(2n+2)(2n+1)} = 0 < 1$

$\to R = \infty$

4a. **Find the Taylor series for f(x) centered at the given value of a.**

$x^5 + 4x^3 + x, a = 3 \to f^n(x) = 0 \text{ for } n \geq 6$

$f(x) = x^5 + 4x^3 + x = \sum_{n=0}^{\infty} \frac{f^n(3)}{n!}(x-3)^n$

$= \frac{354}{0!}(x-3)^0 + \frac{514}{1!}(x-3)^1 + \frac{612}{2!}(x-3)^2 + \frac{564}{3!}(x-3)^3$

$+ \frac{360}{4!}(x-3)^4 + (x-3)^5$

4b. **Find the associated radius of convergence R.**

Finite series converges for all x; $R = \infty$