Integration by Parts — Key

Problem 1. $\int xe^x\,dx$ (Hint: $u = x,\ dv = e^x dx$.)

Solution. Choose $u = x$ and $dv = e^x dx$. Then $du = dx$, $v = e^x$. By integration by parts,

$$\int xe^x\,dx = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

$$\boxed{\int xe^x\,dx = e^x(x-1) + C.}$$

Problem 2. $\int x\cos x\,dx$ (Hint: $u = x,\ dv = \cos x\,dx$.)

Solution. Let $u = x,\ dv = \cos x\,dx$. Then $du = dx,\ v = \sin x$. Integration by parts:

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

$$\boxed{\int x\cos x\,dx = x\sin x + \cos x + C.}$$

Problem 3. $\int x^2\cos(3x)\,dx$.

Solution. Use integration by parts twice.
First integration by parts: choose $u = x^2,\ dv = \cos(3x)\,dx$. Then $du = 2x\,dx$, $v = \frac{1}{3}\sin(3x)$. Thus

$$\int x^2 \cos(3x)\,dx = \frac{x^2}{3}\sin(3x) - \frac{2}{3}\int x\sin(3x)\,dx.$$

Compute $\int x\sin(3x)\,dx$ by parts: let $u = x,\ dv = \sin(3x)\,dx$. Then $du = dx,\ v = -\frac{1}{3}\cos(3x)$. So

$$\int x\sin(3x)\,dx = -\frac{x}{3}\cos(3x) + \frac{1}{9}\sin(3x) + C.$$

Substitute back:

$$\int x^2\cos(3x)\,dx = \frac{x^2}{3}\sin(3x) - \frac{2}{3}\Big(- \frac{x}{3}\cos(3x) + \frac{1}{9}\sin(3x)\Big) + C$$

$$= \frac{x^2}{3}\sin(3x) + \frac{2}{9}x\cos(3x) - \frac{2}{27}\sin(3x) + C.$$

Combine the $\sin(3x)$-terms if desired:

$$\boxed{\int x^2\cos(3x)\,dx = \left(\frac{9x^2-2}{27}\right)\sin(3x) + \frac{2}{9}x\cos(3x) + C.}$$

Problem 4. $\int x^2 e^{2x}\,dx$.

Solution. Apply integration by parts twice.
First: $u = x^2,\ dv = e^{2x}dx \Rightarrow du = 2x\,dx,\ v = \frac{1}{2}e^{2x}$. Then

$$\int x^2 e^{2x}\,dx = \frac{x^2}{2}e^{2x} - \int xe^{2x}\,dx.$$

Next compute $\int xe^{2x}\,dx$ by parts: $u = x,\ dv = e^{2x}dx \Rightarrow du = dx,\ v = \frac{1}{2}e^{2x}$. So

$$\int xe^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C.$$

Substitute:

$$\int x^2e^{2x}\,dx = \frac{x^2}{2}e^{2x} - \left(\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right) + C = e^{2x}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) + C.$$

$$\boxed{\int x^2e^{2x}\,dx = e^{2x}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) + C.}$$

Problem 5. $\int \ln(x+3)\,dx$.

Solution. Use integration by parts with $u = \ln(x+3),\ dv = dx$. Then $du = \dfrac{1}{x+3}dx,\ v = x$. So

$$\int \ln(x+3)\,dx = x\ln(x+3) - \int \frac{x}{x+3}\,dx.$$

Simplify the integrand:

$$\frac{x}{x+3} = 1 - \frac{3}{x+3}, \quad \text{so} \quad \int \frac{x}{x+3}\,dx = x - 3\ln|x+3| + C.$$

Therefore

$$\int \ln(x+3)\,dx = x\ln(x+3) - \big(x - 3\ln|x+3|\big) + C = (x+3)\ln|x+3| - x + C.$$

$$\boxed{\int \ln(x+3)\,dx = (x+3)\ln|x+3| - x + C.}$$

Problem 6. $\displaystyle\int 4x\cos(2-3x)\,dx.$

Solution. Use integration by parts with $u = 4x,\ dv = \cos(2-3x)\,dx$. First find v:

$$\int \cos(2-3x)\,dx = -\frac{1}{3}\sin(2-3x) + C.$$

Thus $v = -\frac{1}{3}\sin(2-3x)$, and $du = 4\,dx$.
Integration by parts:

$$\int 4x\cos(2-3x)\,dx = -\frac{4}{3}x\sin(2-3x) + \frac{4}{3}\int \sin(2-3x)\,dx.$$

Compute the remaining integral:

$$\int \sin(2-3x)\,dx = \frac{1}{3}\cos(2-3x) + C.$$

Therefore

$$\int 4x\cos(2-3x)\,dx = -\frac{4}{3}x\sin(2-3x) + \frac{4}{9}\cos(2-3x) + C.$$

$$\boxed{\int 4x\cos(2-3x)\,dx = -\frac{4}{3}x\sin(2-3x) + \frac{4}{9}\cos(2-3x) + C.}$$

Problem 7. $\displaystyle\int x^7\sin(2x^4)\,dx.$

Solution. Use substitution $t = x^4$ (so $dt = 4x^3 dx$). Observe

$$x^7 dx = x^4 \cdot x^3 dx = t \cdot \frac{dt}{4}.$$

Thus

$$\int x^7 \sin(2x^4)\, dx = \frac{1}{4} \int t \sin(2t)\, dt.$$

Now integrate $\int t \sin(2t)\, dt$ by parts: $a = t,\ db = \sin(2t)\, dt$ so $da = dt,\ b = -\frac{1}{2}\cos(2t)$. Then

$$\int t \sin(2t)\, dt = -\frac{t}{2}\cos(2t) + \frac{1}{2}\int \cos(2t)\, dt = -\frac{t}{2}\cos(2t) + \frac{1}{4}\sin(2t) + C.$$

Multiply by $\frac{1}{4}$ and substitute $t = x^4$:

$$\int x^7 \sin(2x^4)\, dx = -\frac{x^4}{8}\cos(2x^4) + \frac{1}{16}\sin(2x^4) + C.$$

$$\boxed{\int x^7 \sin(2x^4)\, dx = -\frac{x^4}{8}\cos(2x^4) + \frac{1}{16}\sin(2x^4) + C.}$$

Problem 8. $\displaystyle\int x^2 e^{4x}\, dx.$

Solution. Integration by parts twice (parallel to Problem 4).
First: $u = x^2,\ dv = e^{4x} dx \Rightarrow du = 2x\, dx,\ v = \frac{1}{4}e^{4x}$. So

$$\int x^2 e^{4x}\, dx = \frac{x^2}{4}e^{4x} - \frac{1}{2}\int x e^{4x}\, dx.$$

Compute $\int x e^{4x}\, dx$ by parts: $u = x,\ dv = e^{4x} dx \Rightarrow du = dx,\ v = \frac{1}{4}e^{4x}$:

$$\int x e^{4x}\, dx = \frac{x}{4}e^{4x} - \frac{1}{16}e^{4x} + C.$$

Substitute back:

$$\int x^2 e^{4x}\, dx = \frac{x^2}{4}e^{4x} - \frac{1}{2}\left(\frac{x}{4}e^{4x} - \frac{1}{16}e^{4x}\right) + C = e^{4x}\left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right) + C.$$

$$\boxed{\int x^2 e^{4x}\, dx = e^{4x}\left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right) + C.}$$